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Existence of Time Periodic Solutions for One-Dimensional Newtonian Filtration Equation with Multiple Delays*

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Abstract

In this paper, we study one-dimensional Newtonian filtration equation including unbounded sources with multiple delays. The existence of nonnegative non-trivial time periodic solutions will be established by the Leray-Schauder fixed point theorem based on some suitable Lyapunov functionals and some a priori estimates for all possible periodic solutions.

Keywords: Newtonian Filtration; Multiple Delays; Periodic Solution.

1 Introduction

Consider the following one-dimensional Newtonian filtration equation with multiple delays

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u^m}{\partial x^2}+au+f(u(x,t-\tau_1),\cdots,u(x,t-\tau_n))+g(x,t)+\gamma\int_{t-\tau_0}^{t}e^{-\alpha(t-s)}u(x,s)ds,$$
$$x\in(0,1),\ t\in\mathbb{R}, \tag{1.1}$$

subject to the homogeneous Dirichlet boundary value condition

$$u(0,t)=u(1,t)=0, \qquad t\in\mathbb{R}, \tag{1.2}$$

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where $m > 1$, a and α are constants, γ is a positive constant, f and g are the known functions satisfying some structure conditions.

This kind of equation arises from a variety of areas in applied mathematics, physics and mathematical ecology. For the case $m = 1$, $n = 1$ and $\gamma = 0$ with $f(r) = \alpha r/(1+r^\beta)$, which appears in the blood cell production model[1]

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - au + f(u(x, t-\tau)) + g(x,t).$$

On the other hand, for the same case, that is $m = 1$, $n = 1$, $\gamma = 0$, with different f, the equation (1.1) also is known as the Hematopoiesis model ($f(r) = e^{-kr}(k > 0)$) as well as Nicholson's blowflies model ($f(r) = re^{-kr}(k > 0)$), see for example [2, 3]. While, it is worth noting that all the above models are linearly diffusive, but if nonlinear diffusion is introduced, the model will be more consistent with biologic phenomena in the real world. However, as far as we know, only a few works are concerned with time periodic solutions for degenerate parabolic equation with delay(s). For example, in [4], the authors investigated the existence of time periodic solutions for p-Laplacian with multiple delays. In [5], the authors studied the existence of periodic solutions for Nicholson's blowflies model with Newtonian diffusion

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u^m}{\partial x^2} - \delta u + pu(x, t-\tau)e^{-au(x,t-\tau)} + g(x,t) + \beta \int_{t-\tau}^{t} e^{-\alpha(t-s)} u(x,s)ds.$$

Nevertheless, in this paper, a more general source will be discussed, which is allowed to be the blood cell production model or other types.

In the present paper, we pay our attention to the existence of nonnegative time periodic solutions for (1.1). It is worth noticing that in the model of [5], the source with delay is a typical but quite special bounded source. However, in this paper, a more general source will be discussed, particularly, the source with delays is allowed to be unbounded, which caused us difficulties in making the maximum norm estimates and some other a priori estimates. On the other hand, the method used in[4] will also not work for the equation we consider, that is the coefficient matrix associated with Lyapunov function depends on solutions of the problem, and therefore the required estimates as did in [4] could not be obtained. So, we must try some other methods. By constructing some suitable Lyapunov functionals, the a priori estimates for all possible periodic solutions, and combining with Leray-Schauder fixed point theorem, we finally establish the existence of time periodic solutions.

The rest of this paper is organized as follows. In Section 2 we introduce some basic assumptions, preliminary lemmas and state the main results of this paper. Section 3 is devoted to investigating the existence of periodic solutions based on the a priori estimates obtained in Section 2 and Leray-Schauder fixed point theorem.

2 Preliminaries and the Main Result

Throughout this paper, we make the following assumptions:

(H_1) $0 \le g \in C(\overline{Q})$, $g(x,t) \not\equiv 0$, $g(x,t+T) = g(x,t)$;

(H_2) $f(0,\cdots,0) = 0$, $f(r_1,\cdots,r_n) \ge 0$, $r_i \ge 0$ $(i = 1,\cdots,n)$ and

$$|f(a_1,\cdots,a_n) - f(b_1,\cdots,b_n)| \le \sum_{i=1}^{n} \beta_i |a_i - b_i|;$$

where T and β_i are positive constants, $Q = (0,1) \times (0,T)$.

Since the equation (1.1) is degenerate parabolic and problem (1.1)–(1.2) usually admits solutions only in some generalized sense. Hence we introduce the following definition.

Definition 2.1 *A function u is said to be a weak solution of the problem (1.1)–(1.2), if $u \in \{w; w \in L^\infty, w^m \in L^\infty(0,T;W_0^{1,2}(0,1)), \frac{\partial w^m}{\partial t} \in L^2(Q)\}$, and for any $\varphi \in C^\infty(\overline{Q})$ with $\varphi(x,0) = \varphi(x,T)$ and $\varphi(0,t) = \varphi(1,t) = 0$, the following integral equality holds*

$$\int_0^T \int_0^1 \left\{ u\varphi_t - \frac{\partial u^m}{\partial x}\frac{\partial \varphi}{\partial x} + au\varphi + f(u(x,t-\tau_1),\cdots,u(x,t-\tau_n))\varphi + g(x,t)\varphi \right.$$
$$\left. + \left(\gamma \int_{t-\tau}^{t} e^{-\alpha(t-s)} u(x,s)ds\right)\varphi \right\} dxdt = 0.$$

Now we state the main result of this paper.

Theorem 2.1 *Suppose that (H_1) and (H_2) hold. Then the problem (1.1)–(1.2) admits at least one nonnegative T-periodic solution.*

To prove the existence of periodic solutions (1.1)–(1.2), let us first consider the regularized problem

$$\begin{aligned}\frac{\partial u_\varepsilon}{\partial t} =& \frac{\partial^2}{\partial x^2}(\varepsilon u_\varepsilon + u_\varepsilon^m) + au_\varepsilon + f(u_\varepsilon(x,t-\tau_1),\cdots,u_\varepsilon(x,t-\tau_n)) \\ & + g(x,t) + \gamma \int_{t-\tau_0}^{t} e^{-\alpha(t-s)} u_\varepsilon ds, \qquad x \in (0,1),\ t \in \mathbb{R},\end{aligned} \tag{2.1}$$

$$u_\varepsilon(0,t) = u_\varepsilon(1,t), \qquad t \in \mathbb{R}, \tag{2.2}$$

$$u_\varepsilon(x,t) = u_\varepsilon(x,t+T), \qquad x \in [0,1],\ t \in \mathbb{R}. \tag{2.3}$$

The desired solution of the problem (1.1)–(1.2) will be obtained by the limit of some subsequence of solutions u_ε of the regularized problem. However, we need first to establish the existence of solutions u_ε, for which, we will make use of the Leray-Schauder fixed point theorem and our efforts center on obtaining the uniformly boundness of u_ε. To this end, we prove the following lemmas.

Lemma 2.1 *Assume that (H_1), (H_2) hold and u is a nonnegative T-periodic solution of the equation*

$$u_t = \frac{\partial^2}{\partial x^2}(\varepsilon u + u^m) + \lambda\Big(au + f(u(x,t-\tau_1),\cdots,u(x,t-\tau_n)) + g(x,t) + \gamma\int_{t-\tau}^{t} e^{-\alpha(t-s)}u(x,s)ds\Big), \tag{2.4}$$

satisfying the boundary value condition (2.2), where $\lambda \in [0,1]$, $0 < \varepsilon < 1$ is a constant which is arbitrary. Then for any $r > 0$, we have

$$\int_0^T\int_0^1 u^{m+r}dxdt \le C_1(m,r),$$

where $C_1(m,r) > 0$ is a constant which depend on m and r.

Proof. Note that, multiplying Eq.(2.4) by u^r and integrating over Q,

$$\begin{aligned}\int_0^T\int_0^1 u_t u^r dxdt =& \int_0^T\int_0^1 \frac{\partial^2}{\partial x^2}(\varepsilon u + u^m)\cdot u^r dxdt + \lambda\left(a\int_0^T\int_0^1 u^{r+1}dxdt\right.\\ &+\int_0^T\int_0^1 f(u(x,t-\tau_1),\cdots,u(x,t-\tau_n))u^r dxdt + \int_0^T\int_0^1 gu^r dxdt\\ &\left.+\int_0^T\int_0^1 u^r\gamma\int_{t-\tau}^t e^{-\alpha(t-s)}u(x,s)dsdxdt\right).\end{aligned}$$

Since u is T-periodic,

$$\int_0^T\int_0^1 \frac{\partial u}{\partial t}u^r dxdt = \frac{1}{r+1}\int_0^T\int_0^1 \frac{\partial u^{r+1}}{\partial t}dxdt = 0,$$

it follows that

$$\begin{aligned}&\int_0^T\int_0^1 \frac{\partial}{\partial x}(\varepsilon u + u^m)\frac{\partial u^r}{\partial x}dxdt\\ =&\lambda a\int_0^T\int_0^1 u^{r+1}dxdt + \lambda\int_0^T\int_0^1 f(u(x,t-\tau_1),\cdots,u(x,t-\tau_n))u^r dxdt\\ &+\lambda\int_0^T\int_0^1 gu^r dxdt + \lambda\int_0^T\int_0^1 u^r\gamma\int_{t-\tau}^t e^{-\alpha(t-s)}u(x,s)dsdxdt\\ \le&|a|\int_0^T\int_0^1 u^{r+1}dxdt + \sum_{i=1}^n \beta_i\int_0^T\int_0^1 u(x,t-\tau_i)u^r dxdt + K\int_0^T\int_0^1 u^r dxdt\\ &+\int_0^T\int_0^1 u^r\gamma\int_{t-\tau}^t e^{-\alpha(t-s)}u(x,s)dsdxdt\end{aligned}$$

$$
\begin{aligned}
\leq &|a|\varepsilon_1\int_0^T\int_0^1 u^{m+r}dxdt+|a|\varepsilon_1^{-(r+1)/(m-1)}T+\sum_{i=1}^n\beta_i\varepsilon_2\int_0^T\int_0^1 u^{m+r}dxdt\\
&+\sum_{i=1}^n\beta_i\varepsilon_2^{-r/m}\varepsilon_3\int_0^T\int_0^1|u(x,t-\tau_i)|^{m+r}dxdt+\sum_{i=1}^n\beta_i\varepsilon_2^{-r/m}\varepsilon_3^{-1/(m-1)}T\\
&+K\varepsilon_4\int_0^T\int_0^1 u^{m+r}dxdt+K\varepsilon_4^{-r/m}T+\int_0^T\int_0^1 u^r\gamma\int_{t-\tau}^t e^{-\alpha(t-s)}u(x,s)dsdxdt\\
\leq &\left(|a|\varepsilon_1+\sum_{i=1}^n\beta_i\varepsilon_2+\sum_{i=1}^n\beta_i\varepsilon_2^{-r/m}\varepsilon_3+K\varepsilon_4\right)\int_0^T\int_0^1 u^{m+r}dxdt\\
&+\left(\varepsilon_1^{-(r+1)/(m-1)}+\sum_{i=1}^n\beta_i\varepsilon_2^{-r/m}\varepsilon_3^{-1/(m-1)}+K\varepsilon_4^{-r/m}\right)T\\
&+\int_0^T\int_0^1 u^r\gamma\int_{t-\tau}^t e^{-\alpha(t-s)}u(x,s)dsdxdt.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
&\int_0^T\int_0^1 u^r\gamma\int_{t-\tau}^t e^{-\alpha(t-s)}u(x,s)dsdxdt\\
\leq &e^{|\alpha|\tau}\gamma\int_0^T\int_0^1 u^r\int_{t-\tau}^t u(x,s)dsdxdt\leq e^{|\alpha|\tau}\gamma\int_0^T\int_0^1 u^r\int_{-\tau}^T u(x,s)dsdxdt\\
\leq &e^{|\alpha|\tau}\gamma\left[\varepsilon_5\int_0^T\int_0^1 u^{m+r}dxdt+\varepsilon_5^{-\frac{r}{m}}\int_0^T\int_0^1\left(\int_{-\tau}^T u(x,s)ds\right)^{\frac{m+r}{m}}dxdt\right]\\
= &e^{|\alpha|\tau}\gamma\left[\varepsilon_5\int_0^T\int_0^1 u^{m+r}dxdt+\varepsilon_5^{-\frac{r}{m}}T\int_0^1\left(\int_{-\tau}^T u(x,s)ds\right)^{\frac{m+r}{m}}dx\right]\\
\leq &e^{|\alpha|\tau}\gamma\left[\varepsilon_5\int_0^T\int_0^1 u^{m+r}dxdt+\varepsilon_5^{-\frac{r}{m}}T(T+\tau)^{\frac{r}{m}}\int_0^1\int_{-\tau}^T u(x,s)^{\frac{m+r}{m}}dsdx\right]\\
\leq &e^{|\alpha|\tau}\gamma\left[\varepsilon_5\int_0^T\int_0^1 u^{m+r}dxdt+\varepsilon_5^{-\frac{r}{m}}T(T+\tau)^{\frac{r}{m}}\left(\left[\frac{\tau}{T}\right]+1\right)\int_0^1\int_0^T u(x,s)^{\frac{m+r}{m}}dsdx\right]\\
\leq &e^{|\alpha|\tau}\gamma\left[\varepsilon_5\int_0^T\int_0^1 u^{m+r}dxdt+\varepsilon_5^{-\frac{r}{m}}T(T+\tau)^{\frac{r}{m}}\left(\left[\frac{\tau}{T}\right]+1\right)\varepsilon_6\int_0^1\int_0^T u(x,s)^{m+r}dsdx\right.\\
&\left.+\varepsilon_5^{-\frac{r}{m}}T^2(T+\tau)^{\frac{r}{m}}\left(\left[\frac{\tau}{T}\right]+1\right)\varepsilon_6^{-\frac{1}{m-1}}\right].
\end{aligned}
$$

Then we know that

$$
\int_0^T\int_0^1\frac{\partial}{\partial x}(\varepsilon u+u^m)\frac{\partial u^r}{\partial x}dxdt
$$

$$
\begin{aligned}
&\le \left(|a|\varepsilon_1 + \sum_{i=1}^{n}\beta_i\varepsilon_2 + \sum_{i=1}^{n}\beta_i\varepsilon_2^{-r/m}\varepsilon_3 + K\varepsilon_4\right)\int_0^T\int_0^1 u^{m+r}dxdt \\
&\quad + \left(\varepsilon_1^{-(r+1)/(m-1)} + \sum_{i=1}^{n}\beta_i\varepsilon_2^{-r/m}\varepsilon_3^{-1/(m-1)} + K\varepsilon_4^{-r/m}\right)T \\
&\quad + \int_0^T\int_0^1 u^r\gamma\int_{t-\tau}^{t} e^{-\alpha(t-s)}u(x,s)dsdxdt \\
&\le \Bigg(|a|\varepsilon_1 + \sum_{i=1}^{n}\beta_i\varepsilon_2 + \sum_{i=1}^{n}\beta_i\varepsilon_2^{-r/m}\varepsilon_3 + K\varepsilon_4 + e^{|\alpha|\tau}\gamma\varepsilon_5 \\
&\quad + e^{|\alpha|\tau}\gamma\varepsilon_5^{-r/m}T(T+\tau)^{r/m}([\frac{\tau}{T}]+1)\varepsilon_6\Bigg)\int_0^T\int_0^1 u^{m+r}dxdt \\
&\quad + \Bigg(\varepsilon_1^{-(r+1)/(m-1)} + \sum_{i=1}^{n}\beta_i\varepsilon_2^{-r/m}\varepsilon_3^{-1/(m-1)} + K\varepsilon_4^{-r/m} \\
&\quad + e^{|\alpha|\tau}\gamma\varepsilon_5^{-\frac{r}{m}}T(T+\tau)^{\frac{r}{m}}\left(\left[\frac{\tau}{T}\right]+1\right)\varepsilon_6^{-\frac{1}{m-1}}\Bigg)T \\
&\le \Bigg(|a|\varepsilon_1 + \sum_{i=1}^{n}\beta_i\varepsilon_2 + \sum_{i=1}^{n}\beta_i\varepsilon_2^{-r/m}\varepsilon_3 + K\varepsilon_4 + e^{|\alpha|\tau}\gamma\varepsilon_5 \\
&\quad + e^{|\alpha|\tau}\gamma\varepsilon_5^{-r/m}T(T+\tau)^{r/m}\left(\left[\frac{\tau}{T}\right]+1\right)\varepsilon_6\Bigg)\int_0^T\int_0^1 u^{m+r}dxdt + C.
\end{aligned}
$$

Here and below, we use $C > 0$ to denote different positive constants depending only on the known quantities. In addition, it is easy to see that

$$
\begin{aligned}
\int_0^T\int_0^1 \frac{\partial}{\partial x}(\varepsilon u + u^m)\frac{\partial u^r}{\partial x}dxdt &= \int_0^T\int_0^1 (\varepsilon + mu^{m-1})ru^{r-1}u_x^2dxdt \\
&\ge \int_0^T\int_0^1 mru^{m+r-2}u_x^2dxdt \\
&= \frac{4mr}{(m+r)^2}\int_0^T\int_0^1 \left|\frac{\partial}{\partial x}u^{(m+r)/2}\right|^2 dxdt,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\frac{4mr}{(m+r)^2}\int_0^T\int_0^1 \left|\frac{\partial}{\partial x}u^{(m+r)/2}\right|^2 dxdt \\
&\le \Bigg(|a|\varepsilon_1 + \sum_{i=1}^{n}\beta_i\varepsilon_2 + \sum_{i=1}^{n}\beta_i\varepsilon_2^{-r/m}\varepsilon_3 + K\varepsilon_4 + e^{|\alpha|\tau}\gamma\varepsilon_5 \\
&\quad + e^{|\alpha|\tau}\gamma\varepsilon_5^{-r/m}T(T+\tau)^{r/m}\left(\left[\frac{\tau}{T}\right]+1\right)\varepsilon_6\Bigg)\int_0^T\int_0^1 u^{m+r}dxdt + C
\end{aligned}
$$

$$\leq \left(|a|\varepsilon_1 + \sum_{i=1}^{n} \beta_i \varepsilon_2 + \sum_{i=1}^{n} \beta_i \varepsilon_2^{-r/m} \varepsilon_3 + K\varepsilon_4 + e^{|\alpha|\tau}\gamma\varepsilon_5 \right.$$

$$\left. + e^{|\alpha|\tau}\gamma\varepsilon_5^{-r/m} T(T+\tau)^{r/m} \left(\left[\frac{\tau}{T}\right] + 1 \right) \varepsilon_6 \right) \mu \int_0^T \int_0^1 \left| \frac{\partial}{\partial x} u^{(m+r)/2} \right|^2 dxdt + C.$$

Thence if ε_1, ε_2, ε_3, ε_4, ε_5 and ε_6 are appropriately small, we can get

$$\int_0^T \int_0^1 \left| \frac{\partial}{\partial x} u^{(m+r)/2} \right|^2 dxdt \leq C(m, r) \tag{2.5}$$

Using Poincaré inequality, we see that

$$\int_0^T \int_0^1 u^{m+r} dxdt \leq C_1(m, r),$$

which completes the proof of Lemma 2.1. □

Lemma 2.2 *Assume that (H_1), (H_2) hold and u is a nonnegative T-periodic solution of the equation (2.4) satisfying the boundary value condition (2.2). Then we have*

$$\int_0^T \int_0^1 \left| \frac{\partial u^m}{\partial x} \right|^2 dxdt \leq C,$$

where $C > 0$ is a constant.

Proof. In fact, choosing $r = m$ in (2.5), we obtain

$$\int_0^T \int_0^1 \left| \frac{\partial u^m}{\partial x} \right|^2 dxdt \leq C.$$

□

Lemma 2.3 *Assume that (H_1), (H_2) hold and u is a nonnegative T-periodic solution of the equation (2.4) satisfying the boundary value condition (2.2). Then we have*

$$\int_0^T \int_0^1 \left| \frac{\partial}{\partial x} (\varepsilon u + u^m) \right|^2 dxdt \leq C,$$

where $C > 0$ is a constant.

Proof. The proof is a direct verification. A simple calculation shows

$$\int_0^T \int_0^1 \left| \frac{\partial}{\partial x} (\varepsilon u + u^m) \right|^2 dxdt$$

$$
\begin{aligned}
&=\int_0^T\int_0^1 \frac{\partial}{\partial x}(\varepsilon u+u^m)\frac{\partial}{\partial x}(\varepsilon u+u^m)dxdt\\
&=\int_0^T (\varepsilon u+u^m)(\varepsilon u_x+mu^{m-1}u_x)|_0^1 dt-\int_0^T\int_0^1 \frac{\partial^2}{\partial x^2}(\varepsilon u+u^m)(\varepsilon u+u^m)dxdt\\
&=-\int_0^T\int_0^1 \frac{\partial^2}{\partial x^2}(\varepsilon u+u^m)(\varepsilon u+u^m)dxdt\\
&=\int_0^T\int_0^1 (\varepsilon u+u^m)\left(-u_t+a\lambda u+\lambda f(u(x,t-\tau_1),\cdots,u(x,t-\tau_n))+\lambda g(x,t)\right.\\
&\quad\left.+\lambda\gamma\int_{t-\tau}^t e^{-\alpha(t-s)}u(x,s)ds\right)dxdt\\
&=\int_0^T\int_0^1 \left[-(\varepsilon u+u^m)u_t+a\lambda u(\varepsilon u+u^m)+\lambda f(u(x,t-\tau_1),\cdots,u(x,t-\tau_n))(\varepsilon u+u^m)\right.\\
&\quad\left.+\lambda g(\varepsilon u+u^m)+\lambda\gamma(\varepsilon u+u^m)\int_{t-\tau}^t e^{-\alpha(t-s)}u(x,s)ds\right]dxdt\\
&\le|a|\int_0^T\int_0^1 u(\varepsilon u+u^m)dxdt+\sum_{i=1}^n\beta_i\int_0^T\int_0^1|u(x,t-\tau_i)|(\varepsilon u+u^m)dxdt\\
&\quad+\int_0^T\int_0^1 g(\varepsilon u+u^m)dxdt+\gamma\int_0^T\int_0^1(\varepsilon u+u^m)\int_{t-\tau}^t e^{-\alpha(t-s)}u(x,s)dsdxdt\\
&\le|a|\varepsilon\int_0^T\int_0^1 u^2dxdt+|a|\int_0^T\int_0^1 u^{m+1}dxdt+K\varepsilon\int_0^T\int_0^1 udxdt+K\int_0^T\int_0^1 u^m dxdt\\
&\quad+\sum_{i=1}^n\beta_i\varepsilon\int_0^T\int_0^1 u^2dxdt+\sum_{i=1}^n\beta_i\varepsilon\int_0^T\int_0^1|u(x,t-\tau_i)|^2dxdt+\sum_{i=1}^n\beta_i\int_0^T\int_0^1 u^{2m}dxdt\\
&\quad+\sum_{i=1}^n\beta_i\int_0^T\int_0^1|u(x,t-\tau_i)|^2dxdt+\gamma\varepsilon\int_0^T\int_0^1 u\int_{t-\tau}^t e^{-\alpha(t-s)}u(x,s)dsdxdt\\
&\quad+\gamma\int_0^T\int_0^1 u^m\int_{t-\tau}^t e^{-\alpha(t-s)}u(x,s)dsdxdt\\
&\le C.
\end{aligned}
$$

□

For the convenience of further discussion, we denote

$$u(t):=u(x,t),\ u(t+\theta):=u(x,t+\theta),$$

and have the following result

Lemma 2.4 *Assume that* (H_1), (H_2) *hold and* u *is a nonnegative* T*-periodic solution of the equation (2.4) satisfying the boundary value condition (2.2). Then we have*

$$\|u\|_{L^\infty(Q)}\le C,$$

where $C > 0$ is a constant.

Proof. Define

$$V(t) = \int_0^1 \left(\frac{1}{2}u^2 + \frac{1}{2}\left|\frac{\partial}{\partial x}(\varepsilon u + u^m)\right|^2\right) dx + \sum_{i=1}^n \beta_i^2 \int_{-\tau_i}^0 \int_0^1 u^2(t+\theta) dx d\theta, \qquad (2.6)$$

then by the Cauchy inequality and assumption (H_1), it follows that

$$\begin{aligned}
V'(t) &= \int_0^1 \left[uu_t + \frac{\partial}{\partial x}(\varepsilon u + u^m)\frac{\partial}{\partial t}\left(\frac{\partial}{\partial x}(\varepsilon u + u^m)\right)\right] dx + \sum_{i=1}^n \beta_i^2 \int_0^1 [u^2 - u^2(t-\tau_i)]dx \\
&= \int_0^1 [uu_t - \frac{\partial^2}{\partial x^2}(\varepsilon u + u^m)(\varepsilon + mu^{m-1})u_t]dx + \sum_{i=1}^n \beta_i^2 \int_0^1 [u^2 - u^2(t-\tau_i)]dx \\
&= \int_0^1 u\left[\frac{\partial^2}{\partial x^2}(\varepsilon u + u^m) + \lambda a u + \lambda f(u(x,t-\tau_1),\cdots,u(x,t-\tau_n)) + \lambda g(t,x)\right. \\
&\quad \left. +\lambda\gamma \int_{t-\tau}^t e^{-\alpha(t-s)}u(x,s)ds\right] dx \\
&\quad + \int_0^1 (\varepsilon + mu^{m-1})u_t \left[(-u_t + \lambda a u + \lambda f(u(x,t-\tau_1),\cdots,u(x,t-\tau_n)) + \lambda g(t,x)\right. \\
&\quad \left. +\lambda\gamma \int_{t-\tau}^t e^{-\alpha(t-s)}u(x,s)ds\right] dx + \sum_{i=1}^n \beta_i^2 \int_0^1 [u^2 - u^2(t-\tau_i)]dx \\
&\le \int_0^1 \left[-\frac{\partial}{\partial x}(\varepsilon u + u^m)\frac{\partial u}{\partial x} + |a|u^2 + \sum_{i=1}^n \beta_i u(x,t-\tau_i)u + |u||g(x,t)|\right] dx \\
&\quad + \int_0^1 \left[-\tilde{f}u_t^2 + |a|\tilde{f}|u_t||u| + \tilde{f}\sum_{i=1}^n \beta_i u(t-\tau_i)|u_t| + \tilde{f}|u_t||g|\right] dx \\
&\quad + \sum_{i=1}^n \beta_i^2 \int_0^1 [u^2 - u^2(t-\tau_i)]dx \\
&\quad + \gamma \int_0^1 u \int_{t-\tau}^t e^{-\alpha(t-s)}u(s)dsdx + \gamma \int_0^1 \tilde{f}|u_t| \int_{t-\tau}^t e^{-\alpha(t-s)}u(s)dsdx \\
&\le \int_0^1 \left(-mu^{m-1}\left|\frac{\partial u}{\partial x}\right|^2 + (|a| + \sum_{i=1}^n \beta_i^2)u^2\right) dx \\
&\quad + \int_0^1 \left[\sum_{i=1}^n \beta_i u(t-\tau_i)u + ug - \tilde{f}u_t^2 + |a|\tilde{f}u|u_t| + \sum_{i=1}^n \beta_i \tilde{f}u(t-\tau_i)|u_t| + \tilde{f}|u_t|g\right] dxdt \\
&\quad + \gamma \int_0^1 u \int_{t-\tau}^t e^{-\alpha(t-s)}u(s)dsdx + \gamma \int_0^1 \tilde{f}|u_t| \int_{t-\tau}^t e^{-\alpha(t-s)}u(s)dsdx
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \left(-mu^{m-1} \left|\frac{\partial u}{\partial x}\right|^2 + (|a| + \sum_{i=1}^n \beta_i^2) u^2 \right) dx + \int_0^1 F dx \\
&\quad \left(\text{since} \int_0^1 u^2 dx \le \varepsilon_1 \int_0^1 u^{m+1} dx + \varepsilon_1^{-\frac{2}{(m-1)}} \le \varepsilon_1 \mu \int_0^1 \left|\frac{\partial u^{(m+1)/2}}{\partial x}\right|^2 dx + \varepsilon_1^{-\frac{2}{(m-1)}} \right) \\
&\le - c_1 \left(\int_0^1 u^2 dx + \int_0^1 \left|\frac{\partial u^{(m+1)/2}}{\partial x}\right|^2 dx \right) + (|a| + \sum_{i=1}^n \beta_i^2 + c_1) \varepsilon_1^{-2/(m-1)} + \int_0^1 F dx,
\end{aligned}
$$

where

$$
\begin{aligned}
0 <&\varepsilon_1 < \frac{1}{(|a| + \sum_{i=1}^n \beta_i^2)\mu} \frac{4m}{(m+1)^2}, \\
c_1 &= \frac{1}{1+\varepsilon_1 \mu} \left(\frac{4m}{(m+1)^2} - (|a| + \sum_{i=1}^n \beta_i^2) \varepsilon_1 \mu \right) > 0 \\
\tilde{f} &= \varepsilon + m u^{m-1}, \\
F &= -\tilde{f} u_t^2 + |a| \tilde{f} u |u_t| + \sum_{i=1}^n \beta_i \tilde{f} u(t-\tau_i)|u_t| + \tilde{f}|u_t| g + \sum_{i=1}^n \beta_i u(t-\tau_i) u \\
&\quad + ug + \gamma u \int_{t-\tau}^t e^{-\alpha(t-s)} u(s) ds + \gamma \tilde{f} |u_t| \int_{t-\tau}^t e^{-\alpha(t-s)} u(s) ds.
\end{aligned}
$$

Letting

$$
c_2 = (|a| + \sum_{i=1}^n \beta_i^2 + c_1) \varepsilon_1^{-2/(m-1)},
$$

it is obvious that

$$
V'(t) \le -c_1 \left(\int_0^1 u^2 dx + \int_0^1 \left|\frac{\partial u^{(m+1)/2}}{\partial x}\right|^2 dx \right) + c_2 + \int_0^1 F dx. \tag{2.7}
$$

On the other hand

$$
\begin{aligned}
&\int_0^T \int_0^1 F dx dt \\
\le &\int_0^T \int_0^1 [-\tilde{f} u_t^2 + |a| \tilde{f} u |u_t| + \sum_{i=1}^n \beta_i \tilde{f} u(t-\tau_i)|u_t| + \tilde{f}|u_t| g + \sum_{i=1}^n \beta_i u(t-\tau_i) u + u|g| \\
&+ \gamma u \int_{t-\tau}^t e^{-\alpha(t-s)} u(s) ds + \gamma \tilde{f} u_t \int_{t-\tau}^t e^{-\alpha(t-s)} u(s) ds] dx dt
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_0^T\int_0^1 \Bigg[-\tilde{f}u_t^2+\frac{\varepsilon_2|a|}{2}\tilde{f}u_t^2+\frac{|a|}{2\varepsilon_2}\tilde{f}u^2+\frac{\varepsilon_3}{2}\sum_{i=1}^n\beta_i\tilde{f}u_t^2+\frac{1}{2\varepsilon_3}\sum_{i=1}^n\beta_i\tilde{f}u^2(t-\tau_i)+\frac{\varepsilon_4}{2}\tilde{f}u_t^2\\
&\quad+\frac{1}{2\varepsilon_4}f|g|^2+\sum_{i=1}^n\beta_iu^2(t-\tau_i)+\sum_{i=1}^n\beta_iu^2+u^2+g^2+\gamma\tilde{f}|u_t|\int_{t-\tau}^t e^{-\alpha(t-s)}u(s)ds\Bigg]dxdt\\
&\quad+C\int_0^T\int_0^1 u^{m+1}dxdt+C.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\int_0^T\int_0^1\gamma\tilde{f}|u_t|\int_{t-\tau}^t e^{-\alpha(t-s)}u(s)dsdxdt\\
\leq& e^{|\alpha|\tau}\gamma\int_0^T\int_0^1\tilde{f}|u_t|\int_{t-\tau}^t u(s)dsdxdt\\
\leq& e^{|\alpha|\tau}\gamma\left[\frac{\varepsilon_5}{2}\int_0^T\int_0^1\tilde{f}u_t^2dxdt+\frac{1}{2\varepsilon_5}\int_0^T\int_0^1\tilde{f}\left(\int_{t-\tau}^t u(s)ds\right)^2dxdt\right]\\
\leq& e^{|\alpha|\tau}\gamma\left[\frac{\varepsilon_5}{2}\int_0^T\int_0^1\tilde{f}u_t^2dxdt+\frac{\tau}{2\varepsilon_5}\int_0^T\int_0^1\tilde{f}\int_{t-\tau}^t u^2(s)dsdxdt\right]\\
\leq& e^{|\alpha|\tau}\gamma\left[\frac{\varepsilon_5}{2}\int_0^T\int_0^1\tilde{f}u_t^2dxdt+\frac{\tau}{2\varepsilon_5}\int_0^T\int_0^1\int_{-\tau}^T u^2(s)dsdxdt+\frac{m\tau}{2\varepsilon_5}\int_0^T\int_0^1 u^{m-1}\int_{-\tau}^T u^2(s)dsdxdt\right]\\
\leq& e^{|\alpha|\tau}\gamma\Bigg[\frac{\varepsilon_5}{2}\int_0^T\int_0^1\tilde{f}u_t^2dxdt+\frac{\tau T}{2\varepsilon_5}\left(\left[\frac{\tau}{T}\right]+1\right)\int_0^T\int_0^1 u^2dxdt+\frac{m\tau}{2\varepsilon_5}\int_0^T\int_0^1 u^{2m}dxdt\\
&+\frac{m\tau}{2\varepsilon_5}\int_0^T\int_0^1\left(\int_{-\tau}^t u^2(s)ds\right)^2dxdt\Bigg]\\
\leq& e^{|\alpha|\tau}\gamma\Bigg[\frac{\varepsilon_5}{2}\int_0^T\int_0^1\tilde{f}u_t^2dxdt+\frac{\tau T}{2\varepsilon_5}\left(\left[\frac{\tau}{T}\right]+1\right)\int_0^T\int_0^1 u^2dxdt+\frac{m\tau}{2\varepsilon_5}\int_0^T\int_0^1 u^{2m}dxdt\\
&+\frac{m\tau T}{2\varepsilon_5}(T+\tau)\left(\left[\frac{\tau}{T}\right]+1\right)\int_0^T\int_0^1 u^4dxdt\Bigg],
\end{aligned}
$$

choosing ε_2, ε_3, ε_4, ε_5 small appropriately, we have

$$
\begin{aligned}
&\int_0^T\int_0^T Fdxdt\\
&\leq\iint_Q\Bigg[(-1+\frac{|a|\varepsilon_2}{2}+\frac{\varepsilon_3}{2}\sum_{i=1}^n\beta_i+\frac{\varepsilon_4}{2}+\frac{\varepsilon_5e^{|\alpha|\tau}}{2}\gamma)\tilde{f}u_t^2+\frac{|a|}{2\varepsilon_2}\tilde{f}u^2+\frac{1}{2\varepsilon_3}\sum_{i=1}^n\beta_i\tilde{f}u^2(t-\tau_i)\\
&\quad+\frac{1}{2\varepsilon_4}\tilde{f}|g^2|+\sum_{i=1}^n\beta_iu^2(t-\tau_i)+(\sum_{i=1}^n\beta_i+1)u^2+g^2+e^{|\alpha|\tau}\gamma\frac{\tau T}{2\varepsilon_5}\left(\left[\frac{\tau}{T}\right]+1\right)u^2
\end{aligned}
$$

$$+e^{|\alpha|\tau}\gamma\frac{m\tau}{2\varepsilon_5}u^{2m}+e^{|\alpha|\tau}\gamma\frac{m\tau T}{2\varepsilon_5}(T+\tau)\left(\left[\frac{\tau}{T}\right]+1\right)u^4\Big]dxdt+C\int_0^T\int_0^1 u^{m+1}dxdt+C.$$

Applying Lemma 2.1, $\|u\|_{L^r}\le C$, for any $r>0$, yields

$$\int_0^T\int_0^1 Fdxdt\le C. \tag{2.8}$$

Since $V(t)$ and u are T-periodic and from (2.7), (2.8), we have

$$\begin{aligned}0=&\int_0^T V'(t)dt\\ \le&-c_1\left(\int_0^T\int_0^1 u^2dxdt+\int_0^T\int_0^1\left|\frac{\partial u^{(m+1)/2}}{\partial x}\right|^2dxdt\right)+c_2T+\int_0^T\int_0^1 Fdxdt,\end{aligned}$$

which implies

$$\int_0^T\int_0^1 u^2dxdt\le C,\qquad \int_0^T\int_0^1\left|\frac{\partial u^{(m+1)/2}}{\partial x}\right|^2dxdt\le C.$$

Applying Lemma 2.1 and Lemma 2.3 yields

$$\begin{aligned}&\int_0^T V(t)dt\\ \le&\int_0^T\int_0^1\left(\frac{1}{2}u^2+\frac{1}{2}\left|\frac{\partial}{\partial x}(\varepsilon u+u^m)\right|^2\right)dxdt+\sum_{i=1}^n\beta_i\int_0^T\int_{-\tau_i}^0\int_0^1 u^2(t+\theta)dxd\theta dt\\ \le&C.\end{aligned}$$

Since V is continuous, there exists a $t_0\in[0,T]$ satisfies

$$V(t_0)\le\frac{C}{T}\le C.$$

Hence, if $t_0\le t\le t_0+T$, we obtain

$$\begin{aligned}V(t)=&V(t_0)+\int_{t_0}^t V'(s)\,ds\\ \le&C+\int_{t_0}^t\left[-c_1\int_0^1\left(u^2+\left|\frac{\partial u^{(m+1)/2}}{\partial x}\right|^2\right)dx+c_2+\int_0^1 Fdx\right]ds\\ \le&C+\int_0^T\int_0^1\left[c_1\left(u^2+\left|\frac{\partial u^{(m+1)/2}}{\partial x}\right|^2\right)+F\right]dxdt+c_2T\end{aligned}$$

$$\leq C.$$

A simple calculation yields

$$\int_0^1 \left|\frac{\partial}{\partial x}(\varepsilon u + u^m)\right|^2 dx = \int_0^1 |(\varepsilon + mu^{m-1})u_x|^2 dx$$
$$\geq \int_0^1 |mu^{m-1}u_x|^2 dx = \int_0^1 \left|\frac{\partial u^m}{\partial x}\right|^2 dx.$$

By the definition of V, we see that

$$\int_0^1 \left(u^2 + \left|\frac{\partial u^m}{\partial x}\right|^2\right) dx \leq \int_0^1 \left(u^2 + \left|\frac{\partial}{\partial x}(\varepsilon u + u^m)\right|^2\right) dx \leq 2V(t) \leq C,$$

which implies

$$\sup_{0\leq t\leq T} \int_0^1 \left(u^2 + \left|\frac{\partial u^m}{\partial x}\right|^2\right) dx \leq C.$$

It follows from the definition of u that

$$|u(x,t)|^m = |u^m(x,t)| = \left|u^m(0,t) + \int_0^x \frac{\partial u^m(s,t)}{\partial s} ds\right| = \left|\int_0^x \frac{\partial u^m(s,t)}{\partial s} ds\right|$$
$$\leq \int_0^1 \left|\frac{\partial u^m}{\partial x}\right| dx$$
$$\leq \left(\sup_{0\leq t\leq T} \int_0^1 \left|\frac{\partial u^m}{\partial x}\right|^2 dx\right)^{1/2} \leq C^{1/2},$$

that is

$$\|u\|_{L^\infty(Q)} \leq C^{1/(2m)} := C_0.$$

The proof is complete. $\square$

In addition, we can also easily get the L^2 boundedness of $\frac{\partial u^m}{\partial t}$ as follows.

Lemma 2.5 *Assume that (H_1), (H_2) hold and u is a nonnegative T-periodic solution of the equation (2.1) satisfying the boundary value condition (2.2). Then there exists a constant $C > 0$ such that*

$$\iint_Q \left|\frac{\partial u^m}{\partial t}\right|^2 dxdt \leq C.$$

Proof. Multiplying the equation (2.1) by $\frac{\partial}{\partial t}(\varepsilon u + u^m)$ and integrating over Q, we obtain

$$\iint_Q \frac{\partial u}{\partial t}\frac{\partial}{\partial t}(\varepsilon u + u^m) dxdt$$

$$
\begin{aligned}
=&\iint_Q \frac{\partial^2}{\partial x^2}(\varepsilon u+u^m)\frac{\partial}{\partial t}(\varepsilon u+u^m)dxdt + a\iint_Q u\frac{\partial}{\partial t}(\varepsilon u+u^m)dxdt\\
&+\iint_Q [f(u(x,t-\tau_1),\cdots,u(x,t-\tau_n))+g(x,t)]\frac{\partial}{\partial t}(\varepsilon u+u^m)dxdt\\
&+\iint_Q \gamma\int_{t-\tau}^{t} e^{-\alpha(t-s)}u(s)ds\frac{\partial}{\partial t}(\varepsilon u+u^m)dxdt.
\end{aligned}
$$

A simple calculation yields

$$
\int_0^T\int_0^1 \frac{\partial^2}{\partial x^2}(\varepsilon u+u^m)\frac{\partial}{\partial t}(\varepsilon u+u^m)dxdt = -\frac{1}{2}\int_0^T\int_0^1 \frac{\partial}{\partial t}\left(\frac{\partial}{\partial x}(\varepsilon u+u^m)\right)^2 dxdt = 0.
$$

Therefore, by Lemma 2.4, we obtain

$$
\begin{aligned}
&\varepsilon\int_0^T\int_0^1 u_t^2dxdt + \int_0^T\int_0^1 mu^{m-1}u_t^2dxdt\\
\leq&\sum_{i=1}^{n}\beta_i\int_0^T\int_0^1 u(t-\tau_i)\left|\frac{\partial}{\partial t}(\varepsilon u+u^m)\right|dxdt + \int_0^T\int_0^1 g(x,t)\frac{\partial}{\partial t}(\varepsilon u+u^m)dxdt\\
&+\iint_Q \gamma\left(\int_{t-\tau}^{t} e^{-\alpha(t-s)}u(s)ds\right)\frac{\partial}{\partial t}(\varepsilon u+u^m)dxdt\\
\leq&C\int_0^T\int_0^T\left|\frac{\partial}{\partial t}(\varepsilon u+u^m)\right|dxdt\\
\leq&\frac{\varepsilon^2 C\varepsilon_1}{2}\int_0^T\int_0^1 |u_t|^2dxdt + \frac{C}{2\varepsilon_1} + \frac{C\varepsilon_2}{2}\int_0^T\int_0^1 |mu^{m-1}u_t|^2dxdt + \frac{C}{2\varepsilon_2}\\
\leq&\varepsilon\int_0^T\int_0^1 |u_t|^2dxdt + C + \frac{C\varepsilon_2}{2}\int_0^T\int_0^t mu^{m-1}|u_t|^2dxdt,
\end{aligned}
$$

where $\varepsilon_1 = 2/(C\varepsilon)$. Letting ε_2 be appropriately small,

$$
\int_0^T\int_0^1 mu^{m-1}u_t^2dxdt \leq C,
$$

then we can see

$$
\begin{aligned}
\int_0^T\int_0^1\left|\frac{\partial u^m}{\partial t}\right|^2dxdt =& \int_0^T\int_0^1 m^2u^{2(m-1)}u_t^2dxdt\\
\leq& mC^{m-1}\int_0^T\int_0^1 mu^{m-1}u_t^2dxdt \leq C,
\end{aligned}
$$

which completes the proof of Lemma 2.5. □

We will show the Hölder norm estimate of solutions in the following lemma.

Lemma 2.6 *Assume that* (H_1), (H_2) *hold and* u *is a nonnegative* T*-periodic solution of the equation (2.1) satisfying the boundary value condition (2.2). Then there exists a constant* $C > 0$ *such that* $u \in C^{\alpha,\alpha/2}(Q)$ *with* $0 < \alpha < 1/2$.

Proof. In fact, through a similar discussion of [6] (see Chapter 2), we know that $u \in L^\infty(0,T;H_0^1(0,1))$ and $\frac{\partial u}{\partial t} \in L^2(Q)$. By direct computations, for any $x_1 < x_2 \in (0,1)$, we conclude that

$$\begin{aligned}|u(x_2,t)-u(x_1,t)| =&\left|\int_{x_1}^{x_2}\frac{\partial u(x,t)}{\partial x}dx\right| \le \int_{x_1}^{x_2}\left|\frac{\partial u(x,t)}{\partial x}\right|dx\\ \le&\left(\int_0^1\left|\frac{\partial u}{\partial x}\right|^2dx\right)^{1/2}|x_2-x_1|^{1/2}\\ \le& C|x_2-x_1|^{1/2}.\end{aligned} \tag{2.9}$$

On the other hand, to prove

$$|u(x,t_2)-u(x,t_1)| \le C|t_2-t_1|^{1/4}, \tag{2.10}$$

we need only consider the case that $0 \le x \le \frac{1}{2}$, $\Delta t = t_2 - t_1 > 0$, $(\Delta t)^\alpha \le \frac{1}{4}$, where α is determined. Integrating (2.2) over $(y, y+(\Delta t)^\alpha) \times (t_1,t_2)$ gives

$$\begin{aligned}&\int_y^{y+(\Delta t)^\alpha}[u(z,t_2)-u(z,t_1)]dz\\ =&\int_{t_1}^{t_2}\left[\frac{\partial}{\partial x}(\varepsilon u(y+(\Delta t)^\alpha,s)+u^m(y+(\Delta t)^\alpha,s))-\frac{\partial}{\partial x}(\varepsilon u(y,s)+u^m(y,s))\right]ds\\ &+\int_{t_1}^{t_2}\int_y^{y+(\Delta t)^\alpha}[au(z,s)+f(u(z,s-\tau_1),\cdots,u(z,s-\tau_n))+g(z,s)]dzds\\ &+\int_{t_1}^{t_2}\int_y^{y+(\Delta t)^\alpha}\gamma\int_{s-\tau_0}^{s}e^{-\alpha(s-\sigma)}u(x,s)d\sigma dzds,\end{aligned}$$

i.e.

$$\begin{aligned}&(\Delta t)^\alpha\int_0^1(u(y+\theta(\Delta t)^\alpha,t_2)-u(y+\theta(\Delta t)^\alpha,t_1))d\theta\\ =&\int_{t_1}^{t_2}\left[\frac{\partial}{\partial x}(\varepsilon u(y+(\Delta t)^\alpha,s)+u^m(y+(\Delta t)^\alpha,s))-\frac{\partial}{\partial x}(\varepsilon u(y,s)+u^m(y,s))\right]ds\\ &+\int_{t_1}^{t_2}\int_y^{y+(\Delta t)^\alpha}[au(z,s)+f(u(z,s-\tau_1),\cdots,u(z,s-\tau_n))+g(z,s)]dzds\\ &+\int_{t_1}^{t_2}\int_y^{y+(\Delta t)^\alpha}\gamma\int_{s-\tau_0}^{s}e^{-\alpha(s-\sigma)}u(x,s)d\sigma dzds,\end{aligned}$$

Integrating the above equality with respect to y over $(x, x+(\Delta t)^\alpha)$, we conclude that

$$
\begin{aligned}
&(\Delta t)^\alpha \int_x^{x+(\Delta t)^\alpha} \int_0^1 (u(y+\theta(\Delta t)^\alpha, t_2) - u(y+\theta(\Delta t)^\alpha, t_1)) d\theta dy \\
=&\int_x^{x+(\Delta t)^\alpha} \int_{t_1}^{t_2} \left[\frac{\partial}{\partial x}(\varepsilon u(y+(\Delta t)^\alpha, s) + u^m(y+(\Delta t)^\alpha, s)) - \frac{\partial}{\partial x}(\varepsilon u(y,s) + u^m(y,s))\right] dsdy \\
&+\int_x^{x+(\Delta t)^\alpha} \int_{t_1}^{t_2} \int_y^{y+(\Delta t)^\alpha} [au(z,s) + f(u(z,s-\tau_1), \cdots, u(z,s-\tau_n)) + g(z,s)] dzdsdy \\
&+\int_x^{x+(\Delta t)^\alpha} \int_{t_1}^{t_2} \int_y^{y+(\Delta t)^\alpha} \gamma \int_{s-\tau_0}^{s} e^{-\alpha(s-\sigma)} u(x,s) d\sigma dzdsdy, \\
\leq& C|(\Delta t)^{\alpha+1}|^{1/2} + C(\Delta t)^{2\alpha+1}.
\end{aligned}
$$

Hence, by a simple calculations, we have

$$
|u(x^*, t_2) - u(x^*, t_1)| \leq C(\Delta t) + C(\Delta t)^{(\alpha+1)/2-2\alpha} \leq C|t_2 - t_1|^{1/4},
$$

where $x^* = y^* + \theta^*(\Delta t)^\alpha$, $y^* \in (x, x+(\Delta t)^\alpha)$, $\theta^* \in (0,1)$, and choose $\alpha = 1/6$ specially, from which we see that (2.10) holds. Therefore, by (2.9) and (2.10), we obtain that

$$
\begin{aligned}
&|u(x_2,t_2) - u(x_1,t_1)| \\
\leq&|u(x_2,t_2) - u(x^*,t_2)| + |u(x^*,t_2) - u(x^*,t_1)| + |u(x^*,t_1) - u(x_1,t_1)| \\
\leq&C(|x_2 - x^*|^{1/2} + |t_2 - t_1|^{1/4} + |x^* - x^1|^{1/2}) \\
\leq&C(|x_2 - x_1| + |t_2 - t_1|^{1/2})^{1/2},
\end{aligned}
$$

which completes the proof of Lemma 2.6. □

3 Proof of the Main Result

By means of the above proved lemmas and the Leray-Schauder fixed point theorem, we can obtain the existence of solutions u_ε of the regularized problem as follows.

Proposition 3.1 *Assume that (H_1) and (H_2) hold. Then the regularized problem (2.2)–(2.4) has a nonnegative T-periodic solution.*

Proof. Denote by $C_T(\overline{Q})$ the set of all continuous functions u with the T-periodicity in t. We study the following regularized equation

$$
\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}(\varepsilon u + |u|^{m-1}u) + g(x,t) \tag{3.1}
$$

where $0 \le g \in C_T(\overline{Q})$. We claim that, if the problem (3.1),(2.2),(2.3) has a unique T-periodic solution u, then u must be nonnegative. In fact, multiplying (3.1) by u_- and integrating over Q, we obtain

$$\iint_Q \frac{\partial u}{\partial t} u_- dxdt = \iint_Q \frac{\partial^2}{\partial x^2}(\varepsilon u + |u|^{m-1}u)u_- dxdt + \iint_Q g u_- dxdt,$$

where $u_- = \min\{0, u(x,t),\ (x,t) \in Q\}$. Making use of integration by parts, we have

$$\iint_Q \frac{\partial}{\partial x}(\varepsilon u_- + |u_-|^{m-1}u_-)\frac{\partial u_-}{\partial x} dxdt = \iint_Q g u_- dxdt \le 0.$$

Since

$$\iint_Q \frac{\partial}{\partial x}(\varepsilon u_- + |u_-|^{m-1}u_-)\frac{\partial u_-}{\partial x} dxdt = \iint_Q (\varepsilon + m|u_-|^{m-1}) \left|\frac{\partial u_-}{\partial x}\right|^2 dxdt,$$

then we get

$$\iint_Q |u_-|^{m-1} \left|\frac{\partial u_-}{\partial x}\right|^2 dxdt \le 0.$$

Therefore,

$$u_- = 0, \qquad a.e. \text{ in } Q$$

By the definition of u_-, we see that

$$u \ge 0, \qquad a.e. \text{ in } Q.$$

Consequently, we can rewrite the equation (3.1) as

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}(\varepsilon u + u^m) + g(x,t), \qquad x \in (0,1),\ t \in \mathbb{R}. \tag{3.2}$$

Hence, we know that, if the problem (3.2),(2.2),(2.3) has a T-periodic solution, it must be nonnegative.

Similarly, we have the same consequence for the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}(\varepsilon u + u^m) - \zeta u + g(x,t), \tag{3.3}$$

with the conditions (2.2) and (2.3), where $\zeta \ge 0$.

On the other hand, with an argument similar to [7], we claim that for any $g \in C_T(\overline{Q})$, the problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}(\varepsilon u + u^m) + g(x,t), \qquad x \in (0,1),\ t \in \mathbb{R},$$

$$u(0,t)=u(1,t), \qquad t\in\mathbb{R},$$
$$u(x,t)=u(x,t+T), \qquad x\in(0,1),\ t\in\mathbb{R}.$$

has a unique solution $u\in C^{\alpha,\alpha/2}(\overline{Q})$. By constructing a homotopy, it is easy to obtain that the problem (3.3),(2.2),(2.3) also admits a solution $u\in C^{\alpha,\alpha/2}(\overline{Q})$.

Next, we will obtain the existence of periodic solutions for the regularized problem (2.1),(2.2),(2.3).

In case that $a\geq 0$, we consider the periodic problem of the homotopy equation for regularized problem

$$\frac{\partial u}{\partial t}=\frac{\partial^2}{\partial x^2}(\varepsilon u+u^m)+\lambda G(x,t), \qquad x\in(0,1),\ t\in\mathbb{R}, \tag{3.4}$$
$$u(0,t)=u(1,t)=0, \qquad t\in\mathbb{R}, \tag{3.5}$$

where for any $v(x,t)\in C_T(\overline{Q})$,

$$G(x,t)=av(x,t)+f(v(x,t-\tau_1),\cdots,v(x,t-\tau_n))+g(x,t)+\gamma\int_{t-\tau_0}^{t}e^{-\alpha(t-s)}v(x,s)ds.$$

Then problem (3.4)–(3.5) admits a unique solution $u\in C_T^{\alpha,\alpha/2}(\overline{Q})$. Define the mapping

$$L: C_T(\overline{Q})\times[0,1]\longrightarrow C_T(\overline{Q}),$$
$$(v,\lambda)\longmapsto u.$$

Since $C_T^{\alpha,\alpha/2}(\overline{Q})$ can be compactly embedded into $C_T(\overline{Q})$, L is compact. By Lemma 2.4, we know that for any fixed point u_λ of the mapping L, there is a constant C_0 independent of ε and λ, such that

$$\|u_\lambda\|_{L^\infty}\leq C_0.$$

Then in applying Leray-Schauder's fixed point theorem, we know that the problem (2.1)–(2.3) admits a solution u_ε.

In case that $a<0$, we consider the periodic problem of the homotopy equation for regularized problem

$$\frac{\partial u}{\partial t}=\frac{\partial^2}{\partial x^2}(\varepsilon u+u^m)+au+\lambda G(x,t), \qquad x\in(0,1),\ t\in\mathbb{R}, \tag{3.6}$$
$$u(0,t)=u(1,t)=0, \qquad t\in\mathbb{R}, \tag{3.7}$$

where for any $v(x,t)\in C_T(\overline{Q})$,

$$G(x,t)=f(v(x,t-\tau_1),\cdots,v(x,t-\tau_n))+g(x,t)+\gamma\int_{t-\tau_0}^{t}e^{-\alpha(t-s)}v(x,s)ds.$$

Then problem (3.6)–(3.7) admits a unique solution $u \in C_T^{\alpha,\alpha/2}(\overline{Q})$. The following progress is the same as above case, then we get that the problem (2.1)–(2.3) admits a solution u_ε. $\square$

Now, we turn to the proof of the our main result based on the above lemmas and Proposition 3.1

The Proof of Theorem 2.1. Let $\varepsilon = 1/h$ $(h = 1, 2, \cdots)$ and we note u_h for the solution of the problem (2.2)–(2.4). Clearly, according to Lemma 2.2, Lemma 2.4 and Lemma 2.5,

$$\begin{aligned} &\|u_h\|_{L^\infty(Q)} \le C_0, \\ &\left\| \frac{\partial u_h^m}{\partial x} \right\|_{L^2(Q)} \le C_1 T, \\ &\left\| \frac{\partial u_h^m}{\partial t} \right\|_{L^2(Q)}^2 \le C. \end{aligned}$$

Hence there exists a subsequence $\{u_h\}_{h=1}^\infty$, supposed to be $\{u_h\}_{h=1}^\infty$ itself, and a function $u \in \{u; u \in L^\infty; u^m \in L^\infty(0, T; W_0^{1,2}(0, 1)); \frac{\partial u^m}{\partial t} \in L^2(Q)\}$, such that

$$\begin{aligned} u_h(x, t) \to u(x, t), &\qquad \text{in} \quad L^2(Q), \\ \frac{\partial u_h^m(x, t)}{\partial x} \to \frac{\partial u^m(x, t)}{\partial x}, &\qquad \text{in} \quad L^2(Q), \\ f(u_h(x, t - \tau_1), \cdots, u_h(x, t - \tau_n)) \to f(u(x, t - \tau_1), \cdots, u(x, t - \tau_n)), &\qquad \text{in} \quad L^2(Q). \end{aligned}$$

Letting $h \to \infty$ in

$$\begin{aligned} \int_0^T \int_0^1 \Big\{ & u_h \varphi_t + \frac{1}{h} u_h \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial u_h^m}{\partial x} \frac{\partial \varphi}{\partial x} + a u_h \varphi + f(u_h(x, t - \tau_1), \cdots, u_h(x, t - \tau_n)) \varphi \\ & + g(x, t) \varphi + \left(\gamma \int_{t-\tau}^t e^{-\alpha(t-s)} u_h(x, s) ds \right) \varphi \Big\} dx dt = 0, \end{aligned}$$

we have

$$\begin{aligned} \int_0^T \int_0^1 \Big\{ & u \varphi_t - \frac{\partial u^m}{\partial x} \frac{\partial \varphi}{\partial x} + a u \varphi + f(u(x, t - \tau_1), \cdots, u(x, t - \tau_n)) \varphi \\ & + g(x, t) \varphi + \left(\gamma \int_{t-\tau}^t e^{-\alpha(t-s)} u(x, s) ds \right) \varphi \Big\} dx dt = 0. \end{aligned}$$

That shows u satisfies the integral identity in the definition of weak solutions. Therefore, the problem (1.1)–(1.2) admits a nonnegative T-periodic solution $u \in \{u; u \in L^\infty; u^m \in L^\infty(0, T; W_0^{1,2}(0, 1)); \frac{\partial u^m}{\partial t} \in L^2(Q)\}$.

Since $g(x, t) \not\equiv 0$, we see that the T-periodic solution is nontrivial. The proof of Theorem 2.1 is complete. $\square$

Remark. In equation (1.1), if the delays depend on time, we may construct a similar Lyapunov-type functional

$$V(t) = \int_0^1 \left(\frac{1}{2}u^2 + \frac{1}{2}\left| \frac{\partial}{\partial x}(\varepsilon u + u^m) \right|^2 \right) dx + \sum_{i=1}^{n} \beta_i^2 \int_{-\tau_i(t)}^{0} \int_0^1 u^2(t+\theta)dxd\theta.$$

Under suitable assumptions, by the similar arguments, the corresponding existence of time periodic solution should be established for evolution equations with variable delays.

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